ON TRACTABILITY AND CONGRUENCE DISTRIBUTIVITY [*]

EMIL KISS [a] AND MATTHEW VALERIOTE [b]

[a] Department of Algebra and Number Theory, Eötvös University, 1117 Budapest, Pázmány Péter sétány 1/c, Hungary
e-mail address: ewkiss@cs.elte.hu

[b] Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, L8S 4K1, Canada
e-mail address: matt@math.mcmaster.ca

ABSTRACT. Constraint languages that arise from finite algebras have recently been an object of study, especially in connection with the Dichotomy Conjecture of Feder and Vardi. An important class of algebras are those that generate congruence distributive varieties and included among this class are lattices, and more generally, those algebras that have near-unanimity term operations. An algebra will generate a congruence distributive variety if and only if it has a sequence of ternary term operations, called Jónsson terms, that satisfy certain equations.

We prove that constraint languages consisting of relations that are invariant under a short sequence of Jónsson terms are tractable by showing that such languages have bounded relational width.

1. INTRODUCTION

The Constraint Satisfaction Problem (CSP) provides a framework for expressing a wide class of combinatorial problems. Given an instance of the CSP, the aim is to determine if there is a way to assign values from a fixed domain to the variables of the instance so that each of its constraints is satisfied. While the entire collection of CSPs forms an **NP**-complete class of problems, a number of subclasses have been shown to be tractable (i.e., to lie in **P**). A major focus of research in this area is to determine the subclasses of the CSP that are tractable.

One way to define a subclass of the CSP is to restrict the constraint relations that occur in an instance to a given finite set of relations over a fixed, finite domain, called a constraint language. A central problem is to classify the constraint languages that give rise to tractable subclasses of the CSP. Currently, all constraint languages that have been investigated have been shown to give rise to a subclass of the CSP that is either **NP**-complete or in **P**. It is

conjectured in [10] that this dichotomy holds for all subclasses arising from finite constraint languages.

In some special cases, the conjectured dichotomy has been verified. For example, the work of Schaefer [18] and of Bulatov [6] establish this over domains of sizes 2 and 3 respectively. For constraint languages over larger domains a number of significant results have been obtained [5, 1, 9].

One method for establishing that the subclass of the CSP associated with a finite constraint language is tractable is to establish a type of local consistency property for the instances in the subclass. In [11] Feder and Vardi introduce a notion of the width of a constraint language and show that languages of bounded width give rise to tractable subclasses of the CSP. There is a natural connection between these subclasses of the CSP and definability within Datalog.

In work by Jeavons and his co-authors an approach to classifying the tractable constraint languages via algebraic methods has been proposed and applied with great success [5]. In essence, their work allows one to associate a finite algebraic structure to each constraint language and then to analyze the complexity of the corresponding subclass of the CSP in purely algebraic terms.

In this paper, we employ the algebraic approach to analyzing constraint languages and with it are able to identify a new, general class of tractable constraint languages. These languages arise from finite algebras that generate congruence distributive varieties, or equivalently, that have a sequence of special term operations, called Jónsson terms, that satisfy certain equations. Theorem 4.1 establishes the tractability of these languages by showing that they are of bounded width. Related to our result is the theorem of Jeavons, Cohen, and Cooper in [14] that establishes the tractability of constraint languages that arise from another class of finite algebras that generate congruence distributive varieties. These algebras are equipped with a special term operation called a near unanimity operation. Dalmau [9] provides an alternate proof of their result.

2. Preliminaries

In this section we introduce the necessary terminology and results on the CSP and from universal algebra that will be needed to prove the main result (Theorem 4.1) of this paper.

In the following discussion we will employ standard terminology and notation when dealing with n-tuples and relations over sets. In particular, if $\vec{a}$ is a tuple over the sequence of domains A_i, $1 \leq i \leq n$, (i.e., is a member of $\prod_{1 \leq i \leq n} A_i$) and I is a subset of $\{1, 2, \ldots, n\}$ then $\mathrm{proj}_I(\vec{a})$ denotes the tuple $(a_i : i \in I) \in \prod_{i \in I} A_i$ over the sequence of domains $(A_i : i \in I)$ and is called the restriction (or the projection) of $\vec{a}$ to I. We extend this projection function to arbitrary relations over the A_i. The ith element of the tuple $\vec{a}$ will be denoted by $\vec{a}(i)$.

For R and S binary relations on a set A, we define the relational product of R and S, denoted $R \circ S$, to be the binary relation consisting of all pairs (a, b) for which there is some c with $(a, c) \in R$ and $(c, b) \in S$.

2.1. The Constraint Satisfaction Problem.

Definition 2.1. *An instance of the constraint satisfaction problem is a triple $P = (V, A, \mathcal{C})$ with*

- V a non-empty, finite set of variables,
- A a non-empty, finite set (or domain),
- $\mathcal{C}$ a set of constraints $\{C_1, \dots, C_q\}$ where each C_i is a pair $(\vec{s}_i, R_i)$ with
  - $\vec{s}_i$ a tuple of variables of length m_i, called the scope of C_i, and
  - R_i an m_i-ary relation over A, called the constraint relation of C_i.

Given an instance P of the CSP we wish to answer the following question:

Is there a solution to P, i.e., is there a function $f : V \to A$ such that for each $i \le q$, the m_i-tuple $f(\vec{s}_i) \in R_i$?

We say that two instances of the CSP having the same set of variables and the same domain are equivalent if they have the same set of solutions.

In general, the class of CSPs is **NP**-complete (see [14]), but by restricting the nature of the constraint relations that are allowed to appear in an instance of the CSP, it is possible to find natural subclasses of the CSP that are tractable.

Definition 2.2. *Let A be a domain and Γ a set of finitary relations over A. CSP(Γ) denotes the collection of all instances of the CSP with domain A and with constraint relations coming from Γ. Γ is called the constraint language of the class CSP(Γ).*

Definition 2.3. *Call a finite constraint language Γ tractable if the class of problems CSP(Γ) is tractable (i.e., lies in **P**). If Γ is infinite and each finite subset Γ' of Γ is tractable then we say that Γ is tractable. If the entire class CSP(Γ) is in **P** then we say that Γ is globally tractable.*

*Γ is said to be **NP**-complete if for some finite subset Γ' of Γ, the class of problems CSP(Γ') is **NP**-complete.*

A key problem in this area is to classify the (globally) tractable constraint languages. Note that in this paper we will assume that $\mathbf{P} \neq \mathbf{NP}$. Feder and Vardi [11] conjecture that every finite constraint language is either tractable or is **NP**-complete.

We will find it convenient to extend the above notions of instances of the CSP and constraint languages to a multi-sorted setting. This approach has been used on a number of occasions, in particular in [3].

Definition 2.4. *A multi-sorted instance of the constraint satisfaction problem is a pair $P = (\mathcal{A}, \mathcal{C})$ where*

- $\mathcal{A} = (A_1, A_2, \dots, A_n)$ *is a sequence of finite, non-empty sets, called the domains of P, and*
- $\mathcal{C}$ *is a set of constraints $\{C_1, \dots, C_q\}$ where each C_i is a pair (S_i, R_i) with*
  - S_i *a non-empty subset of $\{1, 2, \dots, n\}$ called the scope of C_i, and*
  - R_i *an $|S_i|$-ary relation over $(A_j : j \in S_i)$, called the constraint relation of C_i.*

In this case, a solution to P is an n-tuple $\vec{a}$ over the sequence $(A_i : 1 \le i \le n)$ such that $\mathrm{proj}_{S_j}(\vec{a}) \in R_j$ for each $1 \le j \le q$. Clearly, each standard instance of the CSP can be expressed as an equivalent multi-sorted instance. While the given definition of a multi-sorted instance of the CSP does not allow for the repetition of variables within the scope of any constraint, there is a natural extension of Definition 2.1 that allows this. Note that there is a very straightforward procedure to transform such an instance to an equivalent one that conforms to Definition 2.4.

Definition 2.5. *A relation R over the sets A_i, $1 \le i \le n$, is subdirect if for all $1 \le i \le n$, $\mathrm{proj}_{\{i\}}(R) = A_i$. We call a multi-sorted instance P of the CSP subdirect if each of its constraint relations is.*

In addition to the set of solutions of an instance of the CSP, one can also consider partial solutions of the instance.

Definition 2.6. *For P as in Definition 2.4 and I a subset of $\{1, 2, \ldots, n\}$, the set of partial solutions of P over I, denoted P_I, is the set of solutions of the instance $P' = (\mathcal{A}', \mathcal{C}')$ where $\mathcal{A}' = (A_i : i \in I)$ and $\mathcal{C} = \{C_1', \ldots, C_q'\}$ with $C_j' = (I \cap S_j, \mathrm{proj}_{(I \cap S_j)}(R_j))$ for $1 \le j \le q$.*

Clearly if the set of partial solutions of an instance over some subset of coordinates is empty then the instance has no solutions.

Definition 2.7. *Let $\mathcal{C}$ be a finite set (or sequence) of finite, non-empty sets. A (multi-sorted) constraint language over $\mathcal{C}$ is a collection of finitary relations over the sets in $\mathcal{C}$. Given a multi-sorted constraint language Γ over $\mathcal{C}$, the class CSP(Γ) consists of all multi-sorted instances of the CSP whose domains come from $\mathcal{C}$ and whose constraint relations come from Γ. $\Gamma_{\mathcal{C}}$ denotes the set of all finitary relations over the members of $\mathcal{C}$.*

In a natural way, the notions of tractability and **NP**-completeness can be extended to multi-sorted constraint languages.

2.2. **Algebras.** There are a number of standard sources for the basics of universal algebra, for example [7] and [17]. The books [12, 8] provide details on the more specialized aspects of the subject that we will use in this paper.

Definition 2.8. *An algebra $\mathbf{A}$ is a pair (A, F) where A is a non-empty set and F is a (possibly infinite) collection of finitary operations on A. The operations in F are called the basic operations of $\mathbf{A}$. A term operation of an algebra $\mathbf{A}$ is a finitary operation on A that can be obtained by repeated compositions of the basic operations of $\mathbf{A}$.*

We assume some familiarity with the standard algebraic operations of taking subalgebras, homomorphic images and cartesian products. Note that in order to sensibly take a homomorphic image of an algebra, or the cartesian product of a set of algebras or to speak of terms and equations of an algebra we need to have some indexing of the basic operations of the algebras. Algebras that have the same indexing are said to be similar (or of the same similarity type).

When necessary, we distinguish between an algebra and its underlying set, or universe. A subuniverse of an algebra (A, F) is a subset of A that is invariant under F. Note that we allow empty subuniverses but not algebras with empty universes.

Definition 2.9. *A variety of algebras is a collection of similar algebras that is closed under the taking of cartesian products, subalgebras and homomorphic images. If $\mathcal{K}$ is a class of similar algebras then $\mathsf{V}(\mathcal{K})$ denotes the smallest variety that contains $\mathcal{K}$.*

Theorem 2.10 (Birkhoff). *A class $\mathcal{V}$ of similar algebras is a variety if and only if $\mathcal{V}$ can be axiomatized by a set of equations.*

It turns out that for a class $\mathcal{K}$ of similar algebras, $\mathsf{V}(\mathcal{K}) = \mathsf{HSP}(\mathcal{K})$, i.e., the class of homomorphic images of subalgebras of cartesian products of members of $\mathcal{K}$.

Definition 2.11. *Let $\mathbf{A}$ be an algebra.*

(1) *An equivalence relation θ on A is a congruence of $\mathbf{A}$ if it is invariant under the basic operations of $\mathbf{A}$.*
(2) *The congruence lattice of $\mathbf{A}$, denoted $\mathrm{Con}\,(\mathbf{A})$, is the lattice of all congruences of $\mathbf{A}$, ordered by inclusion.*
(3) *0_A denotes the congruence relation $\{(a,a) : a \in A\}$ and 1_A denotes the congruence relation $\{(a,b) : a, b \in A\}$, the smallest and largest congruences of the algebra $\mathbf{A}$, respectively.*
(4) *An algebra $\mathbf{A}$ is simple if 0_A and 1_A are its only congruences.*

The congruence lattice of an algebra is a very useful invariant and the types of congruence lattices that can appear in a variety govern many properties of the algebras in the variety. One particularly relevant and important property of congruence lattices is that of distributivity.

Definition 2.12. *An algebra $\mathbf{A}$ is said to be congruence distributive if its congruence lattice satisfies the distributive law for congruence meet and join. A class of algebras is congruence distributive if all of its members are.*

Definition 2.13. *For $k > 0$, we define $CD(k)$ to be the class of all algebras $\mathbf{A}$ that have a sequence of ternary term operations $p_i(x, y, z)$, $0 \leq i \leq k$, that satisfies the identities:*

$$
\begin{aligned}
p_0(x, y, z) &= x \\
p_k(x, y, z) &= z \\
p_i(x, y, x) &= x \text{ for all } i \\
p_i(x, x, y) &= p_{i+1}(x, x, y) \text{ for all } i \text{ even} \\
p_i(x, y, y) &= p_{i+1}(x, y, y) \text{ for all } i \text{ odd}
\end{aligned}
$$

A sequence of term operations of an algebra $\mathbf{A}$ that satisfies the above equations will be referred to as Jónsson terms of $\mathbf{A}$. The following celebrated theorem of Jónsson relates congruence distributivity to the existence of Jónsson terms.

Theorem 2.14 (Jónsson). *An algebra $\mathbf{A}$ generates a congruence distributive variety if and only if there is some $k > 0$ such that $\mathbf{A}$ is in $CD(k)$. In this case, all algebras in $\mathsf{V}(\mathbf{A})$ lie in $CD(k)$.*

Definition 2.15. *For $k > 1$, define $\mathcal{V}_k$ to be the variety of all algebras that have as basic operations a sequence of $k + 1$ ternary operations $p_i(x, y, z)$, for $0 \leq i \leq k$, that satisfy the equations from Definition 2.13.*

Note that an algebra is in $CD(1)$ if and only if it has size 1 and is in $CD(2)$ if and only if it has a majority term operation (i.e., a term operation $m(x, y, z)$ that satisfies the equations $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$).

Some of the main results and conjectures dealing with the CSP can be expressed in terms of Tame Congruence Theory, a deep theory of the local structure of finite algebras developed by Hobby and McKenzie. Details of this theory may be found in [12] or [8]. The connection between the CSP and Tame Congruence Theory was made by Bulatov, Jeavons, and Krokhin [5] and we will touch on it in the next subsection. In this paper we will only introduce some of the basic terminology of the theory and will omit most details.

In Tame Congruence Theory, five local types of behaviour of finite algebras are identified and studied. The five types are, in order:

(1) the unary type,
(2) the affine or vector-space type,
(3) the 2 element Boolean type,
(4) the 2 element lattice type,
(5) the 2 element semi-lattice type.

We say that an algebra $\mathbf{A}$ omits a particular type if, locally, the corresponding type of behaviour does not occur in $\mathbf{A}$. A class of algebras $\mathcal{C}$ is said to omit a particular type if all finite members of $\mathcal{C}$ omit that type.

In [12], chapter 9, characterizations of finite algebras that generate varieties that omit the unary type or both the unary and affine type are given. The characterizations are similar to that given by Jónsson of the congruence distributive varieties. It easily follows from the characterizations that if $\mathbf{A}$ is a finite algebra that generates a congruence distributive variety then the variety omits both the unary and affine types.

To close this subsection we note a special property of the term operations of the algebras in $\mathcal{V}_k$ for all $k > 1$.

Definition 2.16. *An n-ary operation $f(x_1, \ldots, x_n)$ on a set A is idempotent if for all $a \in A$, $f(a, a, \ldots, a) = a$. An algebra is idempotent if all of its term operations are idempotent.*

Note that idempotency is hereditary in the sense that if a function is the composition of some idempotent operations then it too is idempotent. In another sense, if $\mathbf{A}$ is idempotent then all algebras in $\mathsf{V}(\mathbf{A})$ are idempotent, since this condition can be described equationally. Finally, note that Jónsson terms are idempotent and so all algebras in $\mathcal{V}_k$ for $k > 1$ are idempotent.

2.3. **Algebras and the CSP.** The natural duality between sets of relations (constraint languages) over a set A and sets of operations (algebras) on A has been studied by algebraists for some time. Jeavons and his co-authors [13] have shown how this link between constraint languages and algebras can be used to transfer questions about tractability into equivalent questions about algebras. In this subsection we present a concise overview of this connection.

Definition 2.17. *Let A be a non-empty set.*
(1) *Let R be an n-ary relation over A and $f(\bar{x})$ an m-ary function over A for some n, $m \geq 0$. We say that R is invariant under f and that f is a polymorphism of R if for all $\vec{a}_i \in R$, for $1 \leq i \leq m$, the n-tuple $f(\vec{a}_1, \ldots, \vec{a}_m)$, whose i-th coordinate is equal to $f(\vec{a}_1(i), \ldots, \vec{a}_m(i))$, belongs to R.*
(2) *For Γ a set of relations over A, $\mathrm{Pol}\,(\Gamma)$ denotes the set of functions on A that are polymorphisms of all the relations in Γ.*
(3) *For F a set of finitary operations on A, $\mathrm{Inv}(F)$ denotes the set of all finitary relations on A that are invariant under all operations in F.*
(4) *For Γ a constraint language over A, $\langle \Gamma \rangle$ denotes $\mathrm{Inv}(\mathrm{Pol}\,(\Gamma))$ and $\mathbf{A}_\Gamma$ denotes the algebra $(A, \mathrm{Pol}\,(\Gamma))$.*
(5) *For $\mathbf{A} = (A, F)$, an algebra over A, $\Gamma_{\mathbf{A}}$ denotes the constraint language $\mathrm{Inv}(F)$.*
(6) *We call a finite algebra $\mathbf{A}$ tractable (**NP**-complete) if the constraint language $\Gamma_{\mathbf{A}}$ is.*

Note that if $\mathbf{A}$ is an algebra, then $\mathrm{Inv}(\mathbf{A})$ coincides with the set of all subuniverses of finite cartesian powers of $\mathbf{A}$. Sets of relations of the form $\mathrm{Inv}(\Gamma)$ for a set of relations Γ are known as relational clones. Equivalently, a set of relations Λ over a finite set A is a

relational clone if and only if it is closed under definition by primitive positive formulas (or conjunctive queries).

Theorem 2.18. *([13]) Let Γ be a constraint language on a finite set. If Γ is tractable then so is $\langle \Gamma \rangle$. If $\langle \Gamma \rangle$ is **NP**-complete then so is Γ.*

In algebraic terms, Theorem 2.18 states that a constraint language Γ is tractable (or **NP**-complete) if and only if the algebra $\mathbf{A}_\Gamma$ is. So, the problem of characterizing the tractable constraint languages can be reduced to the problem of characterizing the tractable finite algebras. In a further step, Bulatov, Jeavons and Krokhin [5] provide a reduction down to idempotent algebras. For this class of algebras, they propose the following characterization of tractability.

Conjecture 2.19. *Let $\mathbf{A}$ be a finite idempotent algebra. Then $\mathbf{A}$ is tractable if and only if the variety $\mathsf{V}(\mathbf{A})$ omits the unary type.*

They show that when this condition fails, the algebra is **NP**-complete [5]. They also show that if $\mathbf{A}$ is a finite, idempotent algebra then $\mathsf{V}(\mathbf{A})$ omits the unary type if and only if the class $\mathsf{HS}(\mathbf{A})$ does. This conjecture has been verified for a number of large classes of algebras. For example, results of Schaefer [18] and Bulatov [6] provide a verification for algebras whose universes have size 2 and 3 respectively.

As noted in the introduction, one approach to proving the tractability of a constraint language Γ is to apply a notion of local consistency to the instances in $\mathrm{CSP}(\Gamma)$ to determine if the instances have solutions. We present a notion of width, called relational width, developed by Bulatov and Jeavons [4] that, for finite constraint languages, is closely related to the notion of width defined by Feder and Vardi (see [15, 16]). In this paper we will closely follow the presentation of relational width found in [3].

Definition 2.20. *Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a sequence of finite, non-empty sets, let $P = (\mathcal{A}, \mathcal{C})$ be an instance of the CSP and let $k > 0$. We say that P is k-minimal if:*
  (1) *For each subset I of $\{1, 2, \ldots, n\}$ of size at most k, there is some constraint (S, R) in $\mathcal{C}$ such that $I \subseteq S$, and*
  (2) *If (S_1, R_1) and (S_2, R_2) are constraints in $\mathcal{C}$ and $I \subseteq S_1 \cap S_2$ has size at most k then $\mathrm{proj}_I(R_1) = \mathrm{proj}_I(R_2)$.*

It is not hard to show that the second condition of this definition is equivalent to having the set of partial solutions P_I of P equal to $\mathrm{proj}_I(R_i)$ for all subsets I of size at most k and all i with $I \subseteq S_i$.

Proposition 2.21. *Let Γ be a constraint language and $k > 0$. There is a polynomial time algorithm (the k-minimality algorithm) that converts a given instance P from CSP(Γ) into an equivalent k-minimal instance P' from CSP($\langle \Gamma \rangle$). In fact, if the arities of the constraint relations of P are bounded by an integer $m \geq k$ then the arities of the constraint relations of P' are also bounded by m.*

Proof. See the discussion in Section 3.1 of [3]. $\square$

Definition 2.22. *Let Γ be a constraint language and $k > 0$. We say that Γ has relational width k if for every instance P from CSP(Γ), P has a solution if and only if the constraint relations of P', the equivalent k-minimal instance produced by the k-minimality algorithm, are all non-empty.*

Proposition 2.23. *Let Γ be a constraint language and $k > 0$.*

(1) *If an instance P of the CSP has a solution then the constraint relations of all equivalent instances are non-empty.*

(2) *If Γ has relational width k and $\Delta \subseteq \Gamma$ then Δ also has relational width k.*

(3) *If Γ has relational width k then every k-minimal instance P from CSP(Γ) whose constraint relations are non-empty has a solution.*

(4) *If Γ is of finite relational width then it is globally tractable.*

(5) *If every k-minimal instance from $CSP(\langle\Gamma\rangle)$ whose constraint relations are non-empty has a solution then Γ has relational width k and hence is globally tractable.*

(6) *If Γ is finite and $m \geq k$ is an upper bound on the arities of the relations in Γ then Γ has relational width k if every k-minimal instance from $CSP(\langle\Gamma\rangle)$ whose constraint relations are non-empty and have arity $\leq m$ has a solution.*

Proof. Statement (4) follows from Proposition 2.21, since if Γ has relational width k and P is an instance from CSP(Γ) then in order to determine if P has a solution, it suffices to test if P', the equivalent k-minimal instance produced by the k-minimality algorithm, has non-empty constraint relations. Statements (5) and (6) also follows from Proposition 2.21 since the constraint relations of P' belong to $\langle\Gamma\rangle$ and their arities are no bigger than the maximum of k and the arities of the constraint relations of P. $\qquad\square$

In the case where Γ happens to be a relational clone (i.e., $\Gamma = \langle\Gamma\rangle$) it follows from statements (3) and (5) of the previous proposition that Γ has relational width k if and only if every k-minimal instance of CSP(Γ) whose constraint relations are all non-empty has a solution. For the most part, we are interested in this type of constraint language in this paper.

We note that in [15, 16] it is shown that a finite constraint language has bounded relational width if and only if it has bounded width in the sense of Feder-Vardi. The following conjecture is similar to Conjecture 2.19 and was proposed by Larose and Zádori [16] for constraint languages of bounded width.

Conjecture 2.24. Let $\mathbf{A}$ be a finite idempotent algebra. Then $\mathbf{A}$ is of bounded width if and only if $\mathsf{V}(\mathbf{A})$ omits the unary and affine types.

In [16] Larose and Zádori verify one direction of this conjecture, namely that if $\mathsf{V}(\mathbf{A})$ fails to omit the unary or affine types then $\mathbf{A}$ is not of bounded width. Note that in [2], Bulatov proposes a conjecture that is parallel to 2.24. Larose and the second author have noted that, as with the unary type, one need only check in $\mathsf{HS}(\mathbf{A})$ to determine if $\mathsf{V}(\mathbf{A})$ omits the unary and affine types when $\mathbf{A}$ is finite and idempotent (see Corollary 3.2 of [19] for a more general version of this).

The main result of this paper can be regarded as providing some evidence in support of Conjecture 2.24. Theorem 4.1 establishes that if $\mathbf{A}$ is a finite member of $CD(3)$ then any finite constraint language contained in $\Gamma_{\mathbf{A}}$ is of bounded width and hence tractable.

3. Algebras in $CD(3)$

Recall that the variety $\mathcal{V}_3$ consists of all algebras $\mathbf{A}$ having four basic operations $p_i(x, y, z)$, $0 \leq i \leq 3$ that satisfy the equations of Definition 2.13. Since the equations dictate that p_0 and p_3 are projections onto x and z respectively, they will play no role in the analysis of algebras in $CD(3)$.

3.1. **Jónsson ideals.** For $\mathbf{A}$ an algebra in $\mathcal{V}_3$, define $x \cdot y$ to be the binary term operation $p_1(x, y, y)$ of $\mathbf{A}$. Note that the Jónsson equations imply that $x \cdot y = p_2(x, y, y)$ as well. This "multiplication" will play a crucial role in the proof of the main theorem of this paper.

Definition 3.1. *For X a subset of an algebra $\mathbf{B} \in \mathcal{V}_3$ let $J(X)$ be the smallest subuniverse Y of $\mathbf{B}$ containing X and satisfying the following closure property: if x is in Y and $u \in B$ then $u \cdot x$ is also in Y.*

We will call $J(X)$ the Jónsson ideal of $\mathbf{B}$ generated by X. The concept of a Jónsson ideal was developed in [19] for any algebra that generates a congruence distributive variety and was used in that paper to establish some intersection properties of subalgebras that are related to relational width.

Definition 3.2. *A finite algebra $\mathbf{B} \in \mathcal{V}_3$ will be called Jónsson trivial if it has no proper non-empty Jónsson ideals.*

Note that $\mathbf{B}$ is Jónsson trivial if and only if $J(\{b\}) = B$ for all $b \in B$. Also note that if $\mathbf{B}$ is Jónsson trivial then every homomorphic image of it is, as well.

We now define a notion of distance in an algebra that will be applied to Jónsson trivial algebras to establish some useful features of the subalgebras of their cartesian products.

Definition 3.3. *Let $\mathbf{A}$ and $\mathbf{B}$ be arbitrary similar algebras and $\mathbf{S}$ a subdirect subalgebra of $\mathbf{A} \times \mathbf{B}$.*

(1) *Let $S_0 = 0_A$ and S_1 be the relation on A defined by:*

$$(a, c) \in S_1 \iff (a, b), (c, b) \in S \text{ for some } b \in B.$$

(2) *For $k > 0$, let $S_{k+1} = S_k \circ S_1$.*
(3) *For $a, b \in A$, we write $d(a, b) = k$ if the pair (a, b) is in S_k and not in S_{k-1} and will say that the distance between a and b relative to S is k. If no such k exists, $d(a, b)$ is said to be undefined.*
(4) *If $d(a, b)$ is defined for all a and $b \in A$ we say that $\mathbf{A}$ is connected with respect to $\mathbf{S}$.*

Proposition 3.4. *Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{S}$ be as in the definition.*

(1) *For each $k \geq 0$, the relation S_k is a reflexive, symmetric subuniverse of $\mathbf{A}^2$.*
(2) *If $\mathbf{A}$ is an idempotent algebra and $c \in A$ then for any $k \geq 0$, the set of all elements a with $d(a, c) \leq k$ is a subuniverse of $\mathbf{A}$.*
(3) *If $\mathbf{A}$ is a simple algebra then either $d(a, b)$ is undefined for all $a \neq b \in A$ (equivalently $S_1 = 0_A$) or $\mathbf{A}$ is connected with respect to $\mathbf{S}$.*

Proof. The symmetry of S_1 is immediate from its definition and its reflexivity follows from S being subdirect. To see that it is a subuniverse of $\mathbf{A}^2$, let $t(x_1, \ldots, x_n)$ be a term operation of $\mathbf{A}$ and $(a_i, b_i) \in S_1$ for $1 \leq i \leq n$. Then for all i there are $c_i \in B$ with (a_i, c_i) and $(b_i, c_i) \in S$. Applying t to these pairs shows that $(t(\bar{a}), t(\bar{c}))$ and $(t(\bar{b}), t(\bar{c})) \in S$ and so $(t(\bar{a}), t(\bar{b})) \in S_1$. This establishes that S_1 is a subuniverse of $\mathbf{A}^2$. Since the relational product operation preserves the properties of symmetry, reflexivity and being a subuniverse, it follows that S_k has all three properties, for $k \geq 0$.

Suppose that $\mathbf{A}$ is idempotent, $c \in A$, and $k \geq 0$. If $t(x_1, \ldots, x_n)$ is a term operation of $\mathbf{A}$ and $a_i \in A$ with $d(a_i, c) \leq k$, for $1 \leq i \leq n$, then $(a_i, c) \in S_k$ for all i. By the first claim of this proposition, it follows that $(t(a_1, \ldots, a_n), t(c, \ldots, c)) \in S_k$ since S_k is a subuniverse of $\mathbf{A}^2$. By idempotency we have $t(c, \ldots, c) = c$ and so $(t(a_1, \ldots, a_n), c) \in S_k$, or $d(t(a_1, \ldots, a_n), c) \leq k$. This establishes the second claim of the proposition.

For the last claim, note that since S_1 is a symmetric, reflexive subuniverse of $\mathbf{A}^2$ then its transitive closure is a congruence on $\mathbf{A}$ that is equal to the union of the S_k, $k \geq 0$. Since $\mathbf{A}$ is assumed to be simple then this congruence is either 0_A or 1_A. In the former case we conclude that $d(a, b)$ is undefined for all $a \neq b \in A$ and in the latter case that for all a, $b \in A$, $(a, b) \in S_k$ for some $k \geq 0$ and so $d(a, b)$ is defined. $\qquad \square$

Lemma 3.5. *Let $\mathbf{A}$ and $\mathbf{B}$ be finite algebras in $\mathcal{V}_3$ and $\mathbf{S}$ a subdirect subalgebra of $\mathbf{A} \times \mathbf{B}$. Suppose that $\mathbf{A}$ is connected with respect to $\mathbf{S}$. Then for every $x, y, z \in A$ we have*

$$d(x \cdot y, z) \leq \max \left(\left[\frac{d(x, y) + 1}{2} \right], d(y, z) \right) .$$

Proof. Let $d(y, z) = m$, $d(x, y) = n$ and choose elements $a_i \in A$ for $0 \leq i \leq n$ with $x = a_0$, $a_n = y$ and $(a_i, a_{i+1}) \in S_1$ for $0 \leq i < n$. For k the largest integer below $[(n+1)/2]$ we get that $d(x, a_k)$ and $d(a_k, y)$ are both at most k. Therefore if $d = \max(k, m)$, then the pairs $(x, a_k), (y, a_k), (y, z)$ are in S_d, and so

$$(p_2(x, y, y), p_2(a_k, a_k, z)) \in S_d .$$

But $p_2(x, y, y) = x \cdot y$ and $p_2(a_k, a_k, z) = z$, proving the lemma. $\qquad \square$

Corollary 3.6. *For $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{S}$ as in the previous lemma, suppose that $d(a, b) \leq n$ for all a, $b \in A$. Let $m \geq [(n+1)/2]$ be any integer and $c \in A$. Then the set of all elements of A whose distance from c is at most m is a Jónsson ideal of $\mathbf{A}$.*

Proof. As noted earlier the set $I = \{a \in A : d(a, c) \leq m\}$ is a subuniverse of $\mathbf{A}$ since $\mathbf{A}$ is idempotent. We need only show that I is closed under multiplication on the left. So, suppose that $a \in I$ and $u \in A$. Since $d(u, c) \leq n$, we have $d(u \cdot a, c) \leq \max(m, d(a, c)) \leq m$ by the previous lemma. $\qquad \square$

Corollary 3.7. *Let $\mathbf{A}$ and $\mathbf{B}$ be finite members of $\mathcal{V}_3$ such that $\mathbf{A}$ is Jónsson trivial and connected with respect to some subdirect subalgebra $\mathbf{S}$ of $\mathbf{A} \times \mathbf{B}$. Then $d(a, b) \leq 1$ for all a, $b \in A$ (or equivalently, $S_1 = A^2$).*

Proof. Suppose that the maximum distance n between the points of A is at least 2 and that $a, b \in A$ with $d(a, b) = n$. Then m, the largest integer below $[(n+1)/2]$ is less than n. From the previous lemma, the set of all elements $u \in A$ with $d(a, u) \leq m$ is a proper Jónsson ideal of $\mathbf{A}$, contradicting that $\mathbf{A}$ is Jónsson trivial. $\qquad \square$

Lemma 3.8. *Let $\mathbf{A}$, $\mathbf{B}$ be finite members of $\mathcal{V}_3$ with $\mathbf{A}$ Jónsson trivial and simple and let $\mathbf{S}$ be a subdirect subalgebra of $\mathbf{A} \times \mathbf{B}$. Then either $S = A \times B$, or S is the graph of an onto homomorphism from $\mathbf{B}$ to $\mathbf{A}$.*

Proof. As $\mathbf{A}$ is simple, then either $S_1 = 0_A$ or $\mathbf{A}$ is connected with respect to $\mathbf{S}$. In the former case, we conclude that S is the graph of an onto homomorphism from $\mathbf{B}$ to $\mathbf{A}$ and in the latter, it follows from the previous corollary that $S_1 = A^2$.

For $a \in A$, let $B_a = \{b \in B : (a, b) \in S\}$ and choose a with $|B_a|$ maximal. Let I denote the set of those elements x of A for which $B_x = B_a$. To complete the proof we will need to demonstrate that $I = A$ and $B_a = B$. To show that $I = A$ it will suffice to prove that it is a Jónsson ideal of $\mathbf{A}$.

Indeed, let $u \in A$ and $c \in I$ be arbitrary. Then $(u, c) \in S_1$ (since $S_1 = A^2$) and therefore there is a $b \in B$ such that (u, b) and (c, b) are in S. Note that since $c \in I$ then $b \in B_a$. If d is any element of B_a then $c \in I$ implies that $(c, d) \in S$, so we get that

$$(p_2(u, c, c), p_2(b, b, d)) = (u \cdot c, d) \in S.$$

Since this holds for every $d \in B_a$, we conclude that $u \cdot c \in I$. Finally, since S is subdirect it follows that $B_a = B$. $\square$

We apply this lemma to obtain a simple description of subdirect products of finite, simple, Jónsson trivial members of $\mathcal{V}_3$ and then show how to use this description to prove that certain k-minimal instances of the CSP have solutions, when $k \geq 3$.

Lemma 3.9. *Let* $\mathbf{A}_i$, *for* $1 \leq i \leq n$, *be finite members of* $\mathcal{V}_3$ *with* $\mathbf{A}_1$ *Jónsson trivial. Let* $\mathbf{S}$ *be a subdirect product of the* $\mathbf{A}_i$'s *such that for all* $1 < i \leq n$, *the projection of S onto coordinates 1 and i is equal to* $A_1 \times A_i$. *Then* $S = A_1 \times D$, *where* $D = \mathrm{proj}_{\{2 \leq i \leq n\}}(S)$.

Proof. We prove this by induction on n. For $n = 2$, the result follows by our hypotheses. Consider the case $n = 3$ and let $\mathbf{D}$ be the projection of $\mathbf{S}$ onto $\mathbf{A}_2 \times \mathbf{A}_3$. Let $(u, v) \in D$ and let $I_{(u,v)} = \{a \in A_1 : (a, u, v) \in S\}$. Our goal is to show that $I_{(u,v)} = A_1$ and we can accomplish this by showing that it is a non-empty Jónsson ideal. Clearly $I_{(u,v)}$ is a non-empty subuniverse of $\mathbf{A}_1$ since all algebras involved are idempotent.

Let $a \in I_{(u,v)}$, $b \in A_1$ and choose elements $y \in A_3$ and $x \in A_2$ with (b, u, y) and $(a, x, y) \in S$. By our hypotheses, these elements exist. Applying p_2 to these elements, along with (a, u, v), we get the element $(b \cdot a, u, v)$, showing that $b \cdot a \in I_{(u,v)}$. Thus $I_{(u,v)}$ is a Jónsson ideal.

Now, consider the general case and suppose that the result holds for products of fewer than n factors. Let $\mathbf{S}_1 = \mathrm{proj}_{\{1 \leq i < n\}}(S)$ and $\mathbf{S}_2 = \mathrm{proj}_{\{2 \leq i < n\}}(S)$. Then $\mathbf{S}$ is isomorphic to a subdirect product of $\mathbf{A}_1$, $\mathbf{S}_2$ and $\mathbf{A}_n$ and, by induction, $S_1 = A_1 \times S_2$. Then, applying the result with $n = 3$ to this situation, we conclude that $S = A_1 \times D$, as required. $\square$

Corollary 3.10. *Let* $\mathbf{A}_i$ *be finite, simple, Jónsson trivial members of* $\mathcal{V}_3$, *for* $1 \leq i \leq n$, *and let* $\mathbf{S}$ *be a subdirect product of the* $\mathbf{A}_i$'s. *If, for all* $1 \leq i < j \leq n$, *the projection of S onto* $A_i \times A_j$ *is not the graph of a bijection then* $S = \prod_{1 \leq i \leq n} A_i$.

Proof. For $1 \leq i < j \leq n$, we have, by Lemma 3.8 that either the projection of S onto $A_i \times A_j$ is the graph of a bijection between the two factors (since they are both simple) or is the full product. The former case is ruled out by assumption and so we are in a position to apply the previous lemma inductively to reach the desired conclusion. $\square$

Definition 3.11. *A subdirect product* $\mathbf{S}$ *of the algebras* $\mathbf{A}_i$, $1 \leq i \leq n$, *is said to be almost trivial if, after suitably rearranging the coordinates, there is a partition of* $\{1, 2, \ldots, n\}$ *into intervals* I_j, $1 \leq j \leq p$, *such that* $S = \mathrm{proj}_{I_1}(S) \times \cdots \times \mathrm{proj}_{I_p}(S)$ *and, for each j, if* $I_j = \{i : u \leq i \leq v\}$ *then there are bijections* $\pi_i : A_u \to A_i$, *for* $i \in I_j$ *such that* $\mathrm{proj}_{I_j}(S) = \{(a, \pi_{u+1}(a), \ldots, \pi_v(a)) : a \in A_u\}$.

Corollary 3.12. *Let* $\mathbf{A}_i$ *be finite, simple, Jónsson trivial members of* $\mathcal{V}_3$, *for* $1 \leq i \leq n$, *and let* $\mathbf{S}$ *be a subdirect product of the* $\mathbf{A}_i$'s. *Then* $\mathbf{S}$ *is almost trivial.*

Proof. For $1 \leq i, j \leq n$, set $i \sim j$ if $i = j$ or the projection of S onto A_i and A_j is equal to the graph of a bijection between these two factors. In this case, let $\pi_{i,j}$ denote this bijection.

It is not hard to see that $\sim$ is an equivalence relation on the set $\{1, 2, \ldots, n\}$ and, by applying Lemma 3.8, if $i \not\sim j$ then the projection of S onto A_i and A_j is equal to $A_i \times A_j$. By using the bijections $\pi_{i,j}$ and Corollary 3.10 it is elementary to show that $\mathbf{S}$ is indeed almost trivial. $\square$

For $\mathcal{A}$ a finite sequence of finite algebras, $P = (\mathcal{A}, \mathcal{C})$ denotes a multi-sorted instance of the CSP whose domains are the universes of the algebras in $\mathcal{A}$ and whose constraint relations are subuniverses of cartesian products of members from $\mathcal{A}$.

Theorem 3.13. *Let $\mathcal{A}$ be a finite sequence of finite, simple, Jónsson trivial members of $\mathcal{V}_3$ and let $P = (\mathcal{A}, \mathcal{C})$ be a subdirect, k-minimal instance of the CSP for some $k \geq 3$. If the constraint relations of P are all non-empty then P has a solution.*

Definition 3.11 and analogs of Corollary 3.12 and Theorem 3.13 can be found at the end of Section 3.3 in [3]. The proof of Corollary 3.4 given in that paper can be used to prove our Theorem 3.13. As we shall see, this theorem will form the base of the inductive proof of our main result.

3.2. **The reduction to Jónsson trivial algebras.** The goal of this subsection is to show how to reduce a k-minimal instance P of the CSP whose domains all lie in $\mathcal{V}_3$ and whose constraint relations are all non-empty to another k-minimal, subdirect instance P' whose domains are all Jónsson trivial and whose constraint relations are non-empty. In order to accomplish this, we will need to work with a suitably large $k \geq 3$.

To start, let $\mathcal{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_n)$ be a sequence of finite algebras from $\mathcal{V}_3$ and let $M = \max\{|A_i| \ : \ 1 \leq i \leq n\}$. Let $k > 0$ and $P = (\mathcal{A}, \mathcal{C})$ be a k-minimal instance of the CSP with $\mathcal{C}$ consisting of the constraints $C_i = (S_i, R_i)$, $1 \leq i \leq m$. By taking suitable subalgebras of the $\mathbf{A}_i$ we may assume that P is subdirect and, of course, we also assume that the R_i are all non-empty. In addition, k-minimality assures that we may assume that the scope of each constraint of P consists of at least k variables and that no two constraints have the same k-element set as their scopes.

Since P is k-minimal then its system of partial solutions over k-element sets satisfies an important compatibility property. Namely, if I and K are k-element sets of coordinates then $\mathrm{proj}_{(I \cap K)}(P_I) = \mathrm{proj}_{(I \cap K)}(P_K)$. In this section we will denote P_I by $\Lambda(I)$ and call this function the k-system (of partial solutions) determined by P. Since P is subdirect then for all I, $\Lambda(I)$ will be a subdirect product of the algebras $\mathbf{A}_i$, for $i \in I$.

We wish to consider the situation in which some $\mathbf{A}_i$, say $\mathbf{A}_1$, has a proper Jónsson ideal J. The main result of this subsection is that if the scopes of the constraints of P all have size at most k (and hence exactly k), or if $k \geq M^2$ then we can reduce the question of the solvability of P to the solvability of a k-minimal instance with $\mathbf{A}_1$ replaced by J. Doing so will allow us to proceed by induction to reduce our original instance down to one whose domains are all Jónsson trivial.

So, let J be a proper non-empty Jónsson ideal of $\mathbf{A}_1$ and define Λ_J to be the following function on the set of k-element subsets of $\{1, 2, \ldots, n\}$:

- If I is a k-element set that includes 1 then define $\Lambda_J(I)$ to be $\{\vec{a} \in \Lambda(I) \ : \ \vec{a}(1) \in J\}$.
- If $1 \notin I$, define $\Lambda_J(I)$ to be the set of all $\vec{a} \in \Lambda(I)$ such that for all $i \in I$ the restriction of $\vec{a}$ to $I \setminus \{i\}$ can be extended to an element of $\Lambda_J(\{1\} \cup (I \setminus \{i\}))$.

Lemma 3.14. *If $k \geq 3$ then*
1. *$\Lambda_J(I)$ is non-empty for all I and if $1 \in I$ then the projection of $\Lambda_J(I)$ onto the first coordinate is equal to J.*
2. *For I, K, k-element subsets of $\{1, 2, \ldots, n\}$, $\mathrm{proj}_{(I \cap K)}(\Lambda_J(I)) = \mathrm{proj}_{(I \cap K)}(\Lambda_J(K))$.*

Proof. Since P is subdirect then for any k-element set I with $1 \in I$ we have that $\Lambda_J(I)$ is non-empty and projects onto J in the first coordinate.

Let I be some k-element set of coordinates with $1 \notin I$. For ease of notation, we may assume that $I = \{2, 3, \ldots, k, k+1\}$. Let $\vec{a} = (a_1, a_2, a_3, \ldots, a_k)$ be any member of $\Lambda_J(\{1, 2, \ldots, k\})$. We will show that there is some $a_{k+1} \in A_{k+1}$ such that $(a_2, a_3, \ldots, a_{k+1}) \in \Lambda_J(I)$. This will not only show that $\Lambda_J(I)$ is non-empty, but will also allow us to easily establish condition (2) of the lemma.

We construct the element a_{k+1} as follows. Since Λ is the k-system for P then there is some element $u \in A_{k+1}$ such that $(a_2, \ldots, a_k, u) \in \Lambda(I)$. Furthermore, there is some $v \in A_{k+1}$ such that $(a_1, a_3, \ldots, a_k, v) \in \Lambda(\{1, 3, \ldots, k+1\})$ and then some $v' \in A_2$ with $(v', a_3, \ldots, a_k, v) \in \Lambda(I)$. Similarly, there are w and w' with $(a_1, a_2, a_4, \ldots, a_k, w) \in \Lambda\{1, 2, 4, \ldots, k+1\})$ and $(a_2, w', a_4, \ldots, a_k, w) \in \Lambda(I)$. Let $a_{k+1} = p_1(u, v, w) \in A_{k+1}$. By applying p_1 to the tuples $(a_2, \ldots, a_k, u)$, $(v', a_3, \ldots, a_k, v)$ and $(a_2, w', a_4, \ldots, a_k, w)$ we see that the tuple $(a_2, a_3, \ldots, a_{k+1}) \in \Lambda(I)$.

We now need to show that for all $2 \le i \le k+1$ there is some $b \in J$ with

$$(b, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k+1}) \in \Lambda_J(\{1, 2, \ldots, i-1, i+1, \ldots, k+1\}).$$

There are a number of cases to consider.

- If $i = k+1$ then the tuple $(a_2, \ldots, a_k)$ extends to $(a_1, a_2, \ldots, a_k)$, a member of $\Lambda_J(\{1, 2, \ldots, k\})$, as required.
- If $i = 2$: There are $x \in A_1$ and $y \in A_3$ with $(x, a_3, \ldots, a_k, u)$ and $(a_1, y, a_4, \ldots, a_k, w)$ in $\Lambda(\{1, 3, \ldots, k+1\})$. Applying p_1 to these tuples and the tuple $(a_1, a_3, a_4, \ldots, a_k, v)$ (in the second variable) produces the tuple $(x \cdot a_1, a_3, \ldots, a_k, a_{k+1}) \in \Lambda(\{1, 3, \ldots, k+1\})$. Since $a_1 \in J$ and J is a Jónsson ideal, then $x \cdot a_1 \in J$ and so this tuple belongs to $\Lambda_J(\{1, 3, \ldots, k+1\})$, as required.
- If $i = 3$ or $3 < i < k+1$ then small variations of the previous argument will work.

To complete the proof of this lemma we need to establish the compatibility of Λ_J on overlapping elements of its domain. Let I and L be distinct members of the domain of Λ_J with non-empty intersection N and let $i \in I \setminus L$ and $l \in L \setminus I$.

Let $\vec{a} \in \Lambda_J(I)$ and let $\vec{c}$ be the projection of $\vec{a}$ onto the coordinates in N. The restriction of $\vec{a}$ to $I \setminus \{i\}$ extends to an element $\vec{a}' \in \Lambda_J(\{1\} \cup (I \setminus \{i\}))$. Since Λ is the k-system for P, the restriction of $\vec{a}'$ to $\{1\} \cup N$ extends to an element $\vec{b}'$ of $\Lambda(\{1\} \cup (L \setminus \{l\}))$. Note that $\vec{b}'(1) \in J$ and the restriction of $\vec{b}'$ to N is $\vec{c}$. By the first part of this proof, it follows that the restriction of $\vec{b}'$ to $L \setminus \{l\}$ extends to an element $\vec{b}$ of $\Lambda_J(L)$ as required. $\square$

Corollary 3.15. *If all of the constraints of P have scopes of size k then there is a k-minimal instance P_J of the constraint satisfaction problem over the domains $\mathbf{J}$ and the $\mathbf{A}_i$, for $2 \le i \le n$, whose constraint relations are all non-empty and whose solution set is contained in the solution set of P.*

Proof. It follows from our assumptions on the sizes of the scopes of the constraints of P that the constraints can be indexed by the k-element subsets of $\{1, 2, \ldots, n\}$ and that for such a subset I, the constraint C_I is of the form (I, R_I) where R_I is a subdirect product of the algebras $\mathbf{A}_i$, for $i \in I$.

We set P_J to be the instance of the CSP over the domains $\mathbf{J}$ and the $\mathbf{A}_i$, for $2 \le i \le n$, that has, for each k-element subset I of $\{1, 2, \ldots, n\}$, the constraint $C_I' = (I, R_I')$, where $R_I' = \Lambda_J(I)$. It follows by construction and from the previous lemma that P_J is a k-minimal

instance of the CSP whose constraint relations are all non-empty and whose solutions are also solutions of P. $\square$

The previous corollary can be used to establish the tractability of the constraint languages arising from finite members of $\mathcal{V}_3$, while the following lemma will be used to prove that these languages are in fact globally tractable.

Lemma 3.16. *Assume that $k \geq M^2$ and let $C = (S, R)$ be a constraint of P. Then there is a subuniverse R_J of R such that for all k-element subsets I of S, the projection of R onto I is equal to $\Lambda_J(I)$.*

Proof. For K a subset of S and $\vec{a} \in R$, we will say that $\vec{a}$ is reduced over K if for all $(k-1)$-element subsets I of K, the restriction of $\vec{a}$ to I can be extended to an element of $\Lambda_J(\{1\} \cup I)$. We define R_J to be the set of all tuples $\vec{a} \in R$ that are reduced over S. R_J is also equal to all elements $\vec{a}$ of R such that for all k-element subsets I of S, the restriction of $\vec{a}$ to I is in $\Lambda_J(I)$. R_J is naturally a subuniverse of R and so the challenge is to show that it satisfies the conditions of the lemma. Our proof breaks into two cases, depending on whether or not the coordinate 1 is in S.

Suppose that $1 \in S$. We may assume that $S = \{1, 2, \ldots, m\}$ for some $m \leq n$. We need to show that if I is a k element subset of S and $\vec{a} \in \Lambda_J(I)$ then there is some $\vec{b} \in R_J$ whose restriction to I is $\vec{a}$.

First consider the sub-case where $1 \in I$. If $\vec{a} \in \Lambda_J(I)$ then by the k-minimality of P there is some $\vec{b} \in R$ whose restriction to I is $\vec{a}$. Since $\vec{b}(1) = \vec{a}(1) \in J$ it follows that $\vec{b}$ is in R_J, as required.

Now, suppose that $1 \notin I$ and assume that $I = \{2, 3, \ldots, k+1\}$. By the k-minimality of P there is some $\vec{c} \in R$ whose restriction to I is $\vec{a}$. For each $2 \leq i \leq k+1$ there is some $j_i \in J$ and some $\vec{c}_i \in R$ such that $\vec{c}_i(1) = j_i$ and such that the restrictions of $\vec{c}_i$ and $\vec{a}$ to $I \setminus \{i\}$ are the same.

Since $k > |J|$ it follows from the Pigeonhole principle that there are $i \neq l$ with $j_i = j_l$. We may assume that $i = 2$ and $l = 3$ and set $j = j_i$. Define $\vec{b}$ to be $p_1(\vec{c}, \vec{c}_2, \vec{c}_3)$. This element belongs to R and satisfies: $\vec{b}(1) = \vec{c}(1) \cdot j \in J$ and the restriction of $\vec{b}$ to I is $\vec{a}$. To establish this equality over coordinate 2 we make use of the identity $p_1(x, y, x) = x$ and over coordinate 3 $p_1(x, x, y) = x$. Finally, $\vec{b}$ is in R_J since $\vec{b}(1) \in J$.

For the remaining case, assume that $1 \notin S$, say $S = \{2, 3, \ldots, m+1\}$. We will show by induction on s that if $k - 1 \leq s \leq m - 1$, K is a subset of $\{2, 3, \ldots, m+1\}$ of size s and $\vec{a} \in R$ is reduced over K then if $i \in S \setminus K$ there is some $\vec{b} \in R$ that is reduced over $K \cup \{i\}$ and such that $\text{proj}_K(\vec{a}) = \text{proj}_K(\vec{b})$. A consequence of this claim is that for any k-element subset I of S, any element of $\Lambda_J(I)$ can be extended to a member of R_J. From this, the lemma follows.

Lemma 3.14 establishes the base of this induction. Assume the induction hypothesis holds for $k - 1 \leq s < m - 1$ and let K be a subset of $\{2, 3, \ldots, m+1\}$ of size $s + 1$. By symmetry, we may assume that $K = \{2, 3, \ldots, s+2\}$. Let $\vec{a} \in R$ be reduced over K. We will show that there is some $\vec{a}' \in R$ which equals $\vec{a}$ over K and is reduced over $K \cup \{s+3\}$.

By the induction hypothesis, for each $2 \leq i \leq s+2$ there is some $\vec{a}_i \in R$ such that the projections of $\vec{a}$ and $\vec{a}_i$ onto $K \setminus \{i\}$ are the same and $\vec{a}_i$ is reduced over $(K \cup \{s+3\}) \setminus \{i\}$. By the Pigeonhole principle it follows that there is some $a \in A_{s+3}$ and a set Q contained in K of size at least M such that for $i \in Q$, $\vec{a}_i(s+3) = a$.

Let i and l be distinct members of Q and let $\vec{a}'$ be the element $p_1(\vec{a}, \vec{a}_i, \vec{a}_l)$ of R. Note that over the coordinates in K, $\vec{a}'$ and $\vec{a}$ are equal and that at $s+3$, $\vec{a}'$ equals $b \cdot a$, where $b = \vec{a}(s+3)$.

We claim that $\vec{a}'$ is reduced over $K \cup \{s+3\}$. To establish this we need to show that over any subset U of $K \cup \{s+3\}$ of size $k-1$, the restriction to U of $\vec{a}'$ can be extended to a member of $\Lambda_J(\{1\} \cup U)$. When U avoids the coordinate $s+3$ there is nothing to do, since $\vec{a}$ is reduced over K.

So, assume that U contains $s+3$ and let $\vec{d}$ be an extension to some element in $\Lambda(\{1\} \cup U)$ of the restriction of $\vec{a}$ to U. Since for each $v \in Q$ the element $\vec{a}_v$ is reduced over $(K \cup \{s+3\}) \setminus \{v\}$ then there is a member $\vec{c}_v$ of $\Lambda_J(\{1\} \cup U)$ whose restriction to $U \setminus \{v\}$ is equal to the restriction of $\vec{a}_v$ over this set. If there is some $v \in Q \setminus U$ then the element $p_1(\vec{d}, \vec{c}_v, \vec{c}_v) \in \Lambda_J(\{1\} \cup U)$ witnesses that the restriction of $\vec{a}'$ to U can be extended as desired.

If, on the other hand, $Q \subseteq U$ then choose two elements u and v of Q such that $\vec{c}_u(1) = \vec{c}_v(1) \in J$. An application of the Pigeonhole principle ensures the existence of these elements since $|Q| > |J|$. Then, the element $p_1(\vec{d}, \vec{c}_u, \vec{c}_v) \in \Lambda_J(\{1\} \cup U)$ and its restriction to U is equal the restriction of $\vec{a}'$ on U. $\square$

Corollary 3.17. *If $k \geq M^2$ then there is a k-minimal instance P_J of the constraint satisfaction problem over $\mathbf{J}$ and the $\mathbf{A}_i$, for $2 \leq i \leq n$, whose constraint relations are all non-empty and whose solution set is contained in the solution set of P.*

Proof. From the preceding lemma it follows that the instance P_J over the domains $\mathbf{J}$ and the $\mathbf{A}_i$, for $2 \leq i \leq n$, with constraints $C' = (S, R_J)$, for each constraint $C = (S, R)$ of P, is k-minimal and has all of its constraint relations non-empty. Since the constraint relations of P_J are subsets of the corresponding constraint relations of P then the result follows. $\square$

Theorem 3.18. *Let $\mathcal{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_n)$ be a sequence of finite algebras from $\mathcal{V}_3$ and let $P = (\mathcal{A}, \mathcal{C})$ be a k-minimal instance of the CSP whose constraint relations are non-empty. If $k \geq 3$ and the sizes of the scopes of the constraints of P are bounded by k or if $k \geq M^2$, where $M = \max\{|A_i| : 1 \leq i \leq n\}$, then there is a subdirect k-minimal instance P' of the CSP over Jónsson trivial subalgebras of the $\mathbf{A}_i$ such that the constraint relations of P' are non-empty and the solution set of P' is contained in the solution set of P.*

Proof. This theorem is proved by repeated application of Corollaries 3.15 and 3.17. $\square$

3.3. The reduction to simple algebras.

In this subsection we show, for $k \geq 3$, how to reduce a k-minimal instance of the CSP whose domains are Jónsson trivial members of $\mathcal{V}_3$ and whose constraint relations are all non-empty to one which has in addition, domains that are simple algebras. Our development closely follows parts of the proof of Theorem 3.1 in [3].

Definition 3.19. *Let $\mathbf{A}_i$, $1 \leq i \leq m$, be similar algebras and let $\Theta = (\theta_1, \ldots, \theta_m)$ be a sequence of congruences $\theta_i \in \mathrm{Con}\,(\mathbf{A}_i)$.*

(1) *$\prod_{i=1}^m \theta_i$ denotes the congruence on $\prod_{i=1}^m \mathbf{A}_i$ that identifies two m-tuples $\vec{a}$ and $\vec{b}$ if and only if $(a_i, b_i) \in \theta_i$ for all i.*

(2) *If I is a subset of $\{1, 2, \ldots, m\}$ and $\mathbf{R}$ is a subalgebra of $\prod_{i \in I} \mathbf{A}_i$ then $\mathbf{R}/\Theta$ denotes the quotient of $\mathbf{R}$ by the restriction of the congruence $\prod_{i \in I} \theta_i$ to R.*

Let $\mathcal{A} = (\mathbf{A}_1, \ldots, \mathbf{A}_n)$ be a sequence of finite, Jónsson trivial members of $\mathcal{V}_3$ and let $P = (\mathcal{A}, \mathcal{C})$ be a subdirect, k-minimal instance of the CSP whose constraint relations are all non-empty. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ where, for $1 \leq i \leq m$, $C_i = (S_i, R_i)$ for some subset S_i of $\{1, 2, \ldots, n\}$ and some subuniverse R_i of $\prod_{i \in S_i} \mathbf{A}_i$. Suppose that one of the $\mathbf{A}_i$ is not simple, say for $i = 1$, and let θ_1 be a maximal proper congruence of $\mathbf{A}_1$.

Recall that for $I \subseteq \{1, 2, \ldots, n\}$, P_I denotes the set of partial solutions of P over the variables I. If $|I| \leq k$ then since P is k-minimal, P_I is non-empty and is a subdirect subuniverse of $\prod_{i \in I} \mathbf{A}_i$.

Since the algebra $\mathbf{A}_1/\theta_1$ is a simple, Jónsson trivial algebra then it follows by Lemma 3.8 that for $2 \leq i \leq n$, $P_{\{1,i\}}/(\theta_1 \times 0_{A_i})$ is either the graph of a homomorphism π_i from $\mathbf{A}_i$ onto $\mathbf{A}_1/\theta_1$ or is equal to $A_1/\theta_1 \times A_i$. Let W consist of 1 along with the set of all i for which the former holds. For $2 \leq i \leq n$, let θ_i be the kernel of the map π_i if $i \in W$, and 0_{A_i} otherwise.

Let $\Theta = (\theta_1, \ldots, \theta_n)$ and set $P/\Theta = (\mathcal{A}/\Theta, \mathcal{C}/\Theta)$ where $\mathcal{A}/\Theta = (\mathbf{A}_1/\theta_1, \ldots, \mathbf{A}_n/\theta_n)$ and $\mathcal{C}/\Theta$ consists of the constraints $C_i/\Theta = (S_i, R_i/\Theta)$, for $1 \leq i \leq m$.

Note that since P is subdirect and k-minimal then so is P/Θ and that each $\mathbf{A}_i/\theta_i$ is Jónsson trivial, since this property is preserved by taking quotients.

Lemma 3.20. *If the instance P/Θ has a solution, then there is some k-minimal instance $P' = (\mathcal{A}', \mathcal{C}')$ such that*

- *$\mathcal{A}' = (\mathbf{A}'_1, \ldots, \mathbf{A}'_n)$, where for each $1 \leq i \leq n$, $\mathbf{A}'_i$ a subalgebra of $\mathbf{A}_i$.*
- *A'_1 is a proper subset of A_1,*
- *$\mathcal{C}' = \{C'_1, \ldots, C'_m\}$ where, for each $1 \leq i \leq m$, $C'_i = (S_i, R'_i)$ for some non-empty subuniverse R'_i of R_i.*

Hence, any solution of P' is a solution of P.

Proof. Let $(s_1, \ldots, s_n)$ be a solution of P/Θ. We can regard each s_i as a congruence block of θ_i and hence as a subuniverse of $\mathbf{A}_i$. For $i \in W$, define $\mathbf{A}'_i$ to be the subalgebra of $\mathbf{A}_i$ with universe s_i and for $i \notin W$, set $\mathbf{A}'_i = \mathbf{A}_i$. For $1 \leq j \leq m$, let

$$R'_j = R_j \cap \prod_{i \in S_j} A'_i.$$

We now set out to prove that the instance $P' = (\mathcal{A}', \mathcal{C}')$ has the desired properties. Since θ_1 is a proper congruence of $\mathbf{A}_1$ then s_1 is a proper subset of A_1 and so A'_1 is properly contained in A_1. Since $(s_1, \ldots, s_n)$ is a solution to P/Θ it follows that for $1 \leq j \leq m$, R'_j is a non-empty subuniverse of R_j.

We need only verify that P' is k-minimal, so let $1 \leq a < b \leq m$ and I be some subset of $S_a \cap S_b$ of size at most k. To establish that $\mathrm{proj}_I(R'_a) = \mathrm{proj}_I(R'_b)$ it will suffice to show that

$$\mathrm{proj}_I(R'_i) = \mathrm{proj}_I(R_i) \cap \prod_{l \in I} A'_l.$$

for all i, since P is k-minimal.

By the definition of R'_i it is immediate that the relation on the left of the equality sign is contained in that on the right. In the case that $W \cap S_i = \emptyset$ the other inclusion is also clear.

If $W \cap S_i \neq \emptyset$ we have that $\mathrm{proj}_{W \cap S_i}(R_i/\Theta)$ is a subdirect product of simple, Jónsson trivial algebras that are all isomorphic to $\mathbf{A}_1/\theta_1$. Since the projection of this subdirect product onto any two coordinates in $W \cap S_i$ is equal to the graph of a bijection then in fact,

the entire subdirect product is isomorphic to $\mathbf{A}_1/\theta_1$ in a natural way (using the bijections π_i from the definition of W). Then, using Lemma 3.9 and the definition of W (or more precisely, the complement of W), we conclude that R_i/Θ is isomorphic to $\mathbf{A}_1/\theta_1 \times D$, where $D = \text{proj}_{(S_i \setminus W)}(R_i)$.

Now, suppose that $\vec{a} \in \text{proj}_I(R_i) \cap \prod_{l \in I} A'_l$. Then there is some $\vec{b} \in R_i$ with $\text{proj}_I(\vec{b}) = \vec{a}$. If $W \cap I = \emptyset$ then, by the concluding remark of the previous paragraph, $\text{proj}_{(S_i \setminus W)}(\vec{b})$ and hence $\text{proj}_I(\vec{b})$ can be extended to an element of R_i that lies in $\prod_{l \in S_i} A'_l$ (here we use the fact that we have a solution of P/Θ to work with). This establishes that, in this case, $\vec{a} \in \text{proj}_I(R'_i)$.

Finally, suppose that for some w we have $w \in W \cap I$. The vector $\vec{b}$ from R_i that projects onto $\vec{a}$ over I has the property that $\vec{b}(w) \in s_w$ (since $\vec{a}$ does). The structure of R_i/Θ worked out earlier implies that $\vec{b}(l) \in s_l$ for all $l \in W \cap S_i$ since $(s_1, \ldots, s_n)$ is a solution to P/Θ. From this we conclude that $\vec{b} \in R'_i$, as required. $\qquad\square$

4. Proof of the main result

In the preceding section we established techniques for reducing k-minimal instances of the CSP over domains from $\mathcal{V}_3$ to more manageable instances. The following theorem employs these techniques to establish the finite relational width of constraint languages arising from finite algebras in $CD(3)$.

Let $\mathbf{A}$ be a finite algebra in $CD(3)$. Then $\mathbf{A}$ has term operations $p_1(x, y, z)$ and $p_2(x, y, z)$ that satisfy the equations:

$$
\begin{aligned}
p_i(x, y, x) &= x \quad , i = 1, 2 \\
p_1(x, x, y) &= x \\
p_1(x, y, y) &= p_2(x, y, y) \\
p_2(x, x, y) &= y
\end{aligned}
$$

Recall that associated with $\mathbf{A}$ is the constraint language $\Gamma_{\mathbf{A}} = \text{Inv}(\mathbf{A})$, consisting of all relations invariant under the basic operations of $\mathbf{A}$.

Theorem 4.1. *If Γ is a subset of $\Gamma_{\mathbf{A}}$ whose relations all have arity k or less, for some $k \geq 3$, then Γ has relational width k. In any case, if $M = |A|^2$ then $\Gamma_{\mathbf{A}}$ has relational width M.*

Corollary 4.2. *If Γ is a finite subset of $\Gamma_{\mathbf{A}}$ then Γ is tractable and is of bounded width in the sense of Feder-Vardi. Furthermore, $\Gamma_{\mathbf{A}}$ is globally tractable.*

Proof. (of the Theorem) We may assume that $\mathbf{A} = (A, p_0, p_1, p_2, p_3)$, where $p_0(x, y, z) = x$ and $p_3(x, y, z) = z$ for all x, y, $z \in A$ since if we can establish the theorem for this sort of algebra, it will then apply to all algebras with universe A that have the p_i as term operations.

Our assumption on $\mathbf{A}$ places it in the variety $\mathcal{V}_3$ and so the results from the previous section apply. Let Γ be a subset of $\Gamma_{\mathbf{A}}$. If Γ is finite, let k be the maximum of 3 and the arities of the relations in Γ and replace Γ by Γ_k, the set of all relations in $\Gamma_{\mathbf{A}}$ of arity k or less. Establishing relational width k for this enlarged Γ will, of course, be a stronger result. If Γ is not finite, replace it by $\Gamma_{\mathbf{A}}$ and set $k = |A|^2$. We will show that in either case, Γ has relational width k.

From statements (5) or (6) of Proposition 2.23 it will suffice to show that if P is a k-minimal instance of $CSP(\Gamma)$ whose constraint relations are all non-empty then P has a solution. We may express P in the form $(\mathcal{A}, \mathcal{C})$ where $\mathcal{A} = (\mathbf{A}, \mathbf{A}, \ldots, \mathbf{A})$ is a sequence of length n, for some $n > 0$, and where $\mathcal{C}$ is a set of constraints of the form $C = (S, R)$, for some non-empty subset S of $\{1, 2, \ldots, n\}$ and some non-empty subuniverse R of $\mathbf{A}^{|S|}$.

In order to apply the results from the previous section as seamlessly as possible, we enlarge our language Γ to a closely related, but larger, multi-sorted language. Let $\mathcal{H}$ be the set of all quotients of subalgebras of $\mathbf{A}$. Note that $\mathcal{H}$ is finite and all algebras in it have size at most $|A|$. If $\Gamma = \Gamma_k$, replace it with the set of all subuniverses of l-fold products of algebras from $\mathcal{H}$, for all $1 \le l \le k$, and otherwise, replace it by the set of all subuniverses of finite products of algebras from $\mathcal{H}$. In both cases, we have extended our original constraint language. P can now be viewed as a k-minimal instance of $CSP(\Gamma)$, the class of multi-sorted CSPs whose instances have domains from $\mathcal{H}$ and whose constraint relations are from Γ.

We now prove that every k-minimal instance of $CSP(\Gamma)$ whose constraint relations are non-empty has a solution. If this is not so, let Q be a counter-example such that the sum of the sizes of the domains of Q is as small as possible. Note that independent of this size, no domain of Q is bigger than $|A|$ since they all come from $\mathcal{H}$. Also note that Q must be subdirect.

From Theorem 3.18 it follows that all of the domains of Q are Jónsson trivial. Then, from Lemma 3.20 we can deduce that all of the domains of Q are simple. If not, then either there is a proper quotient of Q that is k-minimal and that does not have a solution, or the k-minimal instance produced by the lemma cannot have a solution. In either case, we contradict the minimality of Q. Thus Q is a subdirect, k-minimal instance of $CSP(\Gamma)$ whose domains are all simple and Jónsson trivial and whose constraint relations are all non-empty. From Theorem 3.13 we conclude that in fact Q has a solution. This contradiction completes the proof of the theorem. $\square$

5. CONCLUSION

The main result of this paper establishes that for certain constraint languages Γ that arise from finite algebras that generate congruence distributive varieties, the problem class $CSP(\Gamma)$ is tractable. This class of constraint languages includes those that are compatible with a majority operation but also includes some languages that were not previously known to be tractable.

We feel that the proof techniques employed in this paper may be useful in extending our results to include all constraint languages that arise from finite algebras that generate congruence distributive varieties and perhaps beyond.

Problem 1: Extend the algebraic tools developed to handle algebras in $CD(3)$ to algebras in $CD(n)$ for any $n > 3$. In particular, generalize the notion of a Jónsson ideal to this wider setting.

We note that in [19] some initial success at extending the notion of a Jónsson ideal has been obtained.

The bound on relational width established for the languages addressed in this paper seems to depend on the size of the underlying domain of the language. Nevertheless, we are

not aware of any constraint language that has finite relational width that is not of relational width 3.

Problem 2: For each $n > 3$, produce a constraint language Γ_n that has relational width n and not $n-1$. As a strengthening of this problem, find Γ_n that in addition have compatible near unanimity operations.

6. Acknowledgments

References

[1] Andrei Bulatov. Tractable conservative constraint satisfaction problems. In Phokion G. Kolaitis, editor, *Proceedings of the Eighteenth Annual IEEE Symp. on Logic in Computer Science, LICS 2003*, pages 321–330. IEEE Computer Society Press, June 2003.

[2] Andrei Bulatov. A graph of a relational structure and constraint satisfaction problems. In *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science, 2004*, pages 448–457. IEEE, 2004.

[3] Andrei Bulatov. Combinatorial problems raised from 2-semilattices. *Journal of Algebra*, 298(2):321–339, 2006.

[4] Andrei Bulatov and Peter Jeavons. Algebraic structures in combinatorial problems. submitted for publication.

[5] Andrei Bulatov, Peter Jeavons, and Andrei Krokhin. Classifying the complexity of constraints using finite algebras. *SIAM J. Comput.*, 34(3):720–742 (electronic), 2005.

[6] Andrei A. Bulatov. A dichotomy theorem for constraint satisfaction problems on a 3-element set. *J. ACM*, 53(1):66–120, 2006.

[7] Stanley Burris and H. P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1981.

[8] Matthias Clasen and Matthew Valeriote. Tame congruence theory. In *Lectures on algebraic model theory*, volume 15 of *Fields Inst. Monogr.*, pages 67–111. Amer. Math. Soc., Providence, RI, 2002.

[9] Victor Dalmau. Generalized majority-minority operations are tractable. In Prakash Panangaden, editor, *Proceedings of the Twentieth Annual IEEE Symp. on Logic in Computer Science, LICS 2005*, pages 438–447. IEEE Computer Society Press, June 2005.

[10] Tomás Feder and Moshe Y. Vardi. Monotone monadic snp and constraint satisfaction. In *STOC '93: Proceedings of the twenty-fifth annual ACM symposium on Theory of computing*, pages 612–622, New York, NY, USA, 1993. ACM Press.

[11] Tomás Feder and Moshe Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: a study through Datalog and group theory. *SIAM J. Comput.*, 28(1):57–104 (electronic), 1999.

[12] David Hobby and Ralph McKenzie. *The structure of finite algebras*, volume 76 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1988. Revised edition: 1996.

[13] Peter Jeavons. On the algebraic structure of combinatorial problems. *Theoret. Comput. Sci.*, 200(1-2):185–204, 1998.

[14] Peter Jeavons, David Cohen, and Martin C. Cooper. Constraints, consistency and closure. *Artificial Intelligence*, 101(1-2):251–265, 1998.

[15] Benoit Larose. Some notes on bounded widths. unpublished, 2004.

[16] Benoit Larose and László Zádori. Bounded width problems and algebras. Accepted by Algebra Universalis, 2006.

[17] R. McKenzie, G. McNulty, and W. Taylor. *Algebras, Lattices, Varieties Volume 1*. Wadsworth and Brooks/Cole, Monterey, California, 1987.

[18] Thomas J. Schaefer. The complexity of satisfiability problems. In *Conference Record of the Tenth Annual ACM Symposium on Theory of Computing (San Diego, Calif., 1978)*, pages 216–226. ACM, New York, 1978.

[19] Matthew Valeriote. A subalgebra intersection property for congruence distributive varieties. *Canadian Journal of Mathematics*, accepted for publication, 2006.